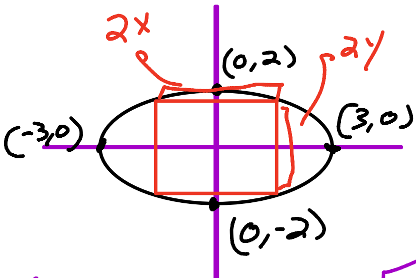
Period 1, April 2, 2025

$$y = \frac{\sqrt{36-4x^2}}{3}$$

13. . What is the area of the largest rectangle that can be inscribed in the ellipse $\frac{4x^2 + 9y^2}{36} = \frac{36}{36}$?

A) $6\sqrt{2}$ B) 12 C) 24 D) $24\sqrt{2}$ E) 36

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$




$$\text{Area} = 2x \cdot 2y = 4xy$$

$$\text{Area} = \frac{4 \cdot x \cdot \sqrt{36-4x^2}}{3}$$

$$\cancel{4x^2} + 9y^2 = 36 - 4x^2$$
$$-4x^2$$

$$\frac{\cancel{9}y^2}{\cancel{9}} = \frac{36-4x^2}{9}$$

$$\sqrt{y^2} = \sqrt{\frac{36-4x^2}{9}}$$

$$y = \frac{\sqrt{36-4x^2}}{3}$$

$$A = \frac{4}{3}x \cdot (36-4x^2)^{\frac{1}{2}}$$

$$\frac{dA}{dx} = \frac{4}{3} \cdot (36-4x^2)^{\frac{1}{2}} + \frac{4}{3}x \cdot \frac{1}{2}(36-4x^2)^{-\frac{1}{2}} \cdot -8x$$

$$\frac{dA}{dx} = 0 = \frac{4\sqrt{36-4x^2}}{3} + \frac{4x \cdot -8x}{6\sqrt{36-4x^2}}$$

$$\sqrt{36-4x^2} \cdot 0 = \left[\frac{4\sqrt{36-4x^2}}{3} + \frac{-16x^2}{3\sqrt{36-4x^2}}\right]\sqrt{36-4x^2}$$

$$0 = \frac{4(36-4x^2)}{3} - \frac{16x^2}{3} = \frac{144-16x^2-16x^2}{3} = 0$$

$$144 - 32x^2 = 0$$

$$\sqrt{\frac{144}{32}} = x$$

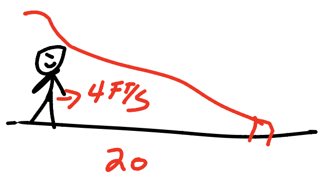
$$x = \frac{12}{4\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

$$A = \frac{4x\sqrt{36-4x^2}}{3}$$

$$12 = 6 \cdot 2 \quad = \frac{6 \cdot \sqrt{2} \cdot 3\sqrt{2}}{\cancel{3}} = \frac{6\sqrt{2} \cdot \sqrt{18}}{3} = \frac{6\sqrt{2}\,\sqrt{36-18}}{3} \Leftarrow \frac{4 \cdot \frac{3\sqrt{2}}{2}\sqrt{36-4\left(\frac{9}{2}\right)}}{3}$$

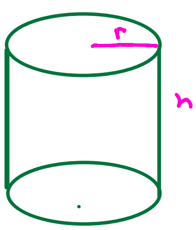
5. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

$\frac{d\theta}{dT} = 0$




12. The volume of a cylindrical tin can with a top and a bottom is to be $16\pi$ cubic inches. If a minimum amount of tin is to be used to construct the can, what must be the height, in inches, of the can?

A) $2\sqrt[3]{2}$ B) $2\sqrt{2}$ C) $2\sqrt[3]{4}$ D) 4 E) 8




Volume $= 16\pi = \pi r^2 h$

$\frac{16\cancel{\pi}}{\cancel{\pi}} = \frac{\cancel{\pi} r^2 h}{\cancel{\pi}}$

$16 = r^2 h$

$\frac{16}{h} = r^2$

$\frac{4}{\sqrt{h}} = r$

$2\pi r h + 2\pi r^2 = SA$

$2\pi \cdot \frac{4}{\sqrt{h}} \cdot h + 2\pi \cdot \frac{16}{h} = SA$

$8\pi\sqrt{h} + 32\pi h^{-1} = SA$

$8\pi h^{\frac{1}{2}} + 32\pi h^{-1} = SA$

$8\pi \cdot \frac{1}{2} h^{-\frac{1}{2}} + 32\pi \cdot -1 h^{-2} = 0$

$\frac{4\pi}{\sqrt{h}} - \frac{32\pi}{h^2} = 0 \Rightarrow \frac{4\pi}{\sqrt{h}} = \frac{32\pi}{h^2}$

$\frac{32\pi\sqrt{h}}{4\pi} = \frac{4\pi h^2}{4\pi}$

$\frac{8\sqrt{h}}{\sqrt{h}} = \frac{h^2}{\sqrt{h}} \Rightarrow 8 = h^{\frac{3}{2}}$

$(8)^{2/3} = (h^{3/2})^{2/3}$

$4 = h$

3. A railroad track and a road cross at right angles. An observer stands on the road 70 meters south of the crossing and watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection?

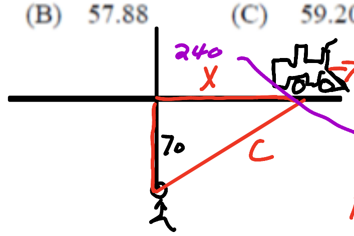
$4 \cdot 60 = 240$ m

(A) 57.60 (B) 57.88 (C) 59.20 (D) 60.00 (E) 67.40

$57.6 = \frac{14400}{250} = \frac{dc}{dT}$

$70^2 + 240^2 = c^2$

$250 = c$




$60\ m/s = \frac{dx}{dT}$

$x^2 + 70^2 = c^2$

$2x\frac{dx}{dT} + 0 = 2c\frac{dc}{dT}$

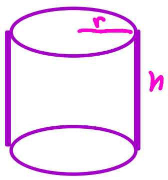
$\cancel{2}(240)60 = \cancel{2} \cdot 250 \cdot \frac{dc}{dT}$

$14400 = 250\frac{dc}{dT}$

14. Consider all right circular cylinders for which the sum of the height and the circumference is 30 centimeters. What is the radius of the one with maximum volume?

A) 3 cm B) 10 cm C) 20 cm D) $\frac{30}{\pi^2}$ cm E) $\frac{10}{\pi}$ cm

$V = \pi r^2 h$




$2\pi r + h = 30$

$h = 30 - 2\pi r$

$V = \pi r^2(30 - 2\pi r)$

$V = 30\pi r^2 - 2\pi^2 r^3$

$\frac{dV}{dr} = 0 = 60\pi r - 6\pi^2 r^2$

$\frac{6\pi^2 r^2}{6\pi^2 r} = \frac{60\pi r}{6\pi^2 r} \Rightarrow r = \frac{10}{\pi}$

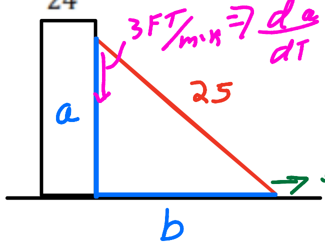
7. The top of a 25-foot ladder is sliding down a vertical wall at a constant rate of 3 feet per minute. When the top of the ladder is 7 feet from the ground, what is the rate of change of the distance between the bottom of the ladder and the wall?

A) ~~$-\frac{7}{8}$ feet per minute~~ D) $\frac{7}{8}$ feet per minute

B) ~~$-\frac{7}{24}$ feet per minute~~ E) $\frac{21}{25}$ feet per minute

C) $\frac{7}{24}$ feet per minute

$a = 7$




$3\,\text{FT}/\text{min} \Rightarrow \frac{da}{dT} = -3\,\text{FT}/\text{min}$

$\frac{db}{dT} = +$

$a^2 + b^2 = 25^2$

$7^2 + b^2 = 25^2$

$b = 24$

$\Rightarrow 2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$

$2(7)(-3) + 2 \cdot 24 \cdot \frac{db}{dT} = 0$

$-42 + 48\frac{db}{dT} = 0$

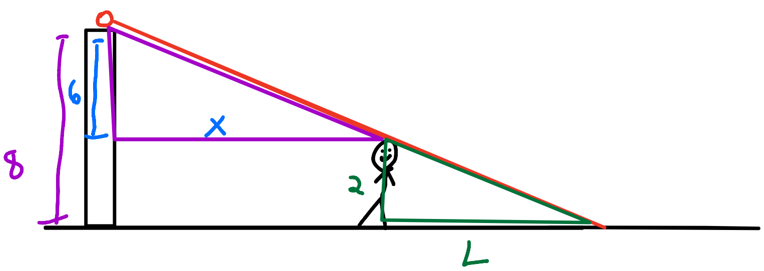
$\frac{48\frac{db}{dT}}{48} = \frac{42}{48} = \frac{7}{8}$

15. A person 2 meters tall walks directly away from a streetlight that is 8 meters above the ground. If the person is walking at a constant rate and the person's shadow is lengthening at the rate of $\frac{4}{9}$ meter per second, at what rate, in meters per second, is the person walking?

$\frac{dx}{dT}$

A) $\frac{4}{27}$ B) $\frac{4}{9}$ C) $\frac{3}{4}$ D) $\frac{4}{3}$ E) $\frac{16}{9}$

$\frac{dL}{dT} = \frac{4}{9}$ m/s




$\frac{6}{x} = \frac{2}{L}$

$2x = 6L$

$2\frac{dx}{dT} = 6\frac{dL}{dT}$

$2\frac{dx}{dT} = 6 \cdot \frac{4}{9} = \frac{24}{9} = \frac{8}{3}$

$\cancel{2}\frac{dx}{dT} = \frac{8}{3 \cdot 2}$

$\frac{dx}{dT} = \frac{8}{6} = \frac{4}{3}$

10. **1998 #3 (BC) - No Calc:** The slope of the line tangent to the curve $y^2 + (xy+1)^3 = 0$ at $(2,-1)$ is

$\frac{dy}{dx}$

a. $\frac{-3}{2}$ c. 0 e. $\frac{3}{2}$

b. $\frac{-3}{4}$ d. $\frac{3}{4}$

$y^2 + (xy+1)^3 = 0$

$2y\frac{dy}{dx} + 3(xy+1)^2 \cdot (y + x\frac{dy}{dx}) = 0$

$2(-1)\frac{dy}{dx} + 3(\underbrace{2(-1)+1}_{-1})^2(-1 + 2\frac{dy}{dx}) = 0$

$-2\frac{dx}{dx} + 3(-1 + 2\frac{dy}{dx}) = 0$

$-2\frac{dy}{dx} - 3 + 6\frac{dy}{dx} = 0 \Rightarrow 4\frac{dy}{dx} = 3 \Rightarrow \frac{3}{4} = \frac{dy}{dx}$

$(xy+1)^3 = m \Rightarrow u^3 = m$

$u = xy+1$

$3u^2\frac{du}{dx} = \frac{dm}{dx}$

$\frac{du}{dx} = 1 \cdot y + x \cdot \frac{dy}{dx} + 0$

$3u^2(y + x\frac{dy}{dx})$

11. **1998 #81 (BC) - Calc OK:** If $\frac{dy}{dx} = \sqrt{1-y^2}$, then $\frac{d^2y}{dx^2} =$

a. $-2y$

b. $-y$

c. $\frac{-y}{\sqrt{1-y^2}}$

d. $y$

e. $\frac{1}{2}$

$\frac{dy}{dx} = (1-y^2)^{\frac{1}{2}}$

$\frac{d^2y}{dx^2} = \frac{1}{2}(1-y^2)^{-\frac{1}{2}}\left(0-2y\frac{dy}{dx}\right)$

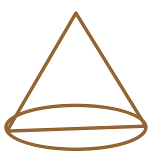
$\frac{d^2y}{dx^2} = \frac{1}{\cancel{2}\cancel{\sqrt{1-y^2}}} \cdot -\cancel{2}y \cdot \cancel{\sqrt{1-y^2}}$

16.

The volume of a cone of radius $r$ and height $h$ is given by $V = \frac{1}{3}\pi r^2 h$. If the radius and the height both increase at a constant rate of $\frac{1}{2}$ centimeter per second, at what rate, in cubic centimeters per second, is the volume increasing when the height is 9 centimeters and the radius is 6 centimeters?

$\frac{dV}{dT} = ?$

A) $\frac{1}{2}\pi$ B) $10\pi$ C) $24\pi$ D) $54\pi$ E) $108\pi$



$V = \frac{1}{3}\pi r^2 h$

$\frac{dr}{dT} = \frac{1}{2}$ cm/s

$\frac{dh}{dT} = \frac{1}{2}$ cm/s

$\frac{dV}{dT} = \frac{2}{3}\pi r \cdot \frac{dr}{dT} \cdot h + \frac{1}{3}\pi r^2 \cdot \frac{dh}{dT}$

$\frac{dV}{dT} = \frac{2\pi}{3} \cdot 6 \cdot \frac{1}{2} \cdot 9 + \frac{1}{3}\pi(6)^2 \cdot \frac{1}{2}$

$\frac{108}{6}\pi + \frac{36\pi}{6} = \frac{144\pi}{6} = 24\pi$

$18\pi + 6\pi = 24\pi$

10 The maximum acceleration attained on the interval $0 \le t \le 3$ by the particle whose velocity is given by $v(t) = t^3 - 3t^2 + 12t + 4$ is D

(A) 9 (B) 12 (C) 14 (D) 21 (E) 40

$V'(T) = a(T) = 3T^2 - 6T + 12$

$V''(T) = a'(T) = 6T - 6$

$0 = 6T - 6$

$T = 1$

| $T$ | $a(T) = 3T^2 - 6T + 12$ |
|---|---|
| 0 | $3(0)^2 - 6(0) + 12 = 12$ |
| 1 | $3(1)^2 - 6(1) + 12 = 9$ |
| 3 | $3(3)^2 - 6(3) + 12$ <br> $27 - 18 + 12 = 21$ |

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8 The function $f$ is given by $f(x) = x^4 + x^2 - 2$. On which of the following intervals is $f$ increasing?

$F'(x) = +$

(A) $\left(-\frac{1}{\sqrt{2}}, \infty\right)$

(B) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(C) $(0, \infty)$

(D) $(-\infty, 0)$

(E) $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$

$F(x) = x^4 + x^2 - 2$

$F'(x) = 4x^3 + 2x$

always be +

$2x(2x^2+1) = 0$

$x = 0$

$- - - - 0 + + + + +$

$-1 \quad 0 \quad 1$

$F(-1) = 4(-1)^3 + 2(-1) = -6$

$F(1) = 4(1)^3 + 2(1) = 6$

$x \geq 0$

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always Positive

$2x^2 + 1 = 0$

$-1 \quad -1$

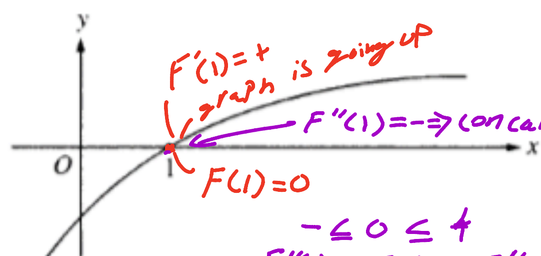
$\frac{2x^2}{2} = \frac{-1}{2}$

$\sqrt{x^2} = \sqrt{-\frac{1}{2}}$

$x = \pm\sqrt{-\frac{1}{2}}$

6

$F'(1) = +$
$F''(1) = -$
$F(1) = 0$




$F'(1) = +$ graph is going up

$F''(1) = - \Rightarrow$ concave Down

$F(1) = 0$

$- \leq 0 \leq +$
$F''(1) \leq F(1) \leq F'(1)$

The graph of a twice-differentiable function $f$ is shown in the figure above. Which of the following is true?

(A) $f(1) < f'(1) < f''(1)$

(B) $f(1) < f''(1) < f'(1)$

(C) $f'(1) < f(1) < f''(1)$

(D) $f''(1) < f(1) < f'(1)$

(E) $f''(1) < f'(1) < f(1)$

3 What are all values of $x$ for which the function $f$ defined by $f(x) = (x^2 - 3)e^{-x}$ is increasing?

(A) There are no such values of $x$.

(B) $x < -1$ and $x > 3$

(C) $-3 < x < 1$

(D) $-1 < x < 3$

(E) All values of $x$

$F'(x) = 2x \cdot e^{-x} + (x^2-3) \cdot e^{-x} \cdot -1$

$\frac{1}{e^x}[2x - x^2 + 3]$

$e^x \cdot 0 = \left[\frac{2x}{e^x} - \frac{x^2-3}{e^x}\right] \cdot e^x$

$e^x$ is always +

$0 = 2x - x^2 + 3$

$x^2 - 2x - 3 = 0$

4 What is the $x$-coordinate of the point of inflection on the graph of $y = \frac{1}{3}x^3 + 5x^2 + 24$? D

(A) 5 (B) 0 (C) $-\frac{10}{3}$ (D) $-5$ (E) $-10$

$F''(x)$ - - - - + + + +

-6 -5 -4

$y = \frac{1}{3}x^3 + 5x^2 + 24$

$\frac{dy}{dx} = \frac{1}{3} \cdot 3x^2 + 5 \cdot 2x^1 = x^2 + 10x$

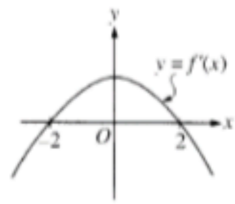
$\frac{dy^2}{d^2x} = 2x + 10 = F''(x)$

$x = -5$

$F''(-6) = 2(-6) + 10 = -2$

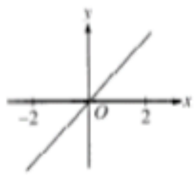
$F''(-4) = 2(-4) + 10 = 2$

2

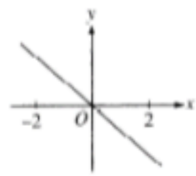



The graph of the derivative of $f$ is shown in the figure above. Which of the following could be the graph of $f$?

(A)

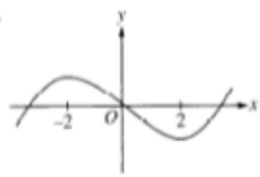



(B)

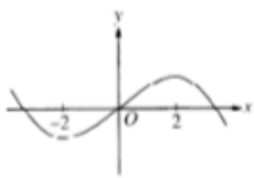



(C)


(D)




(E)



# 2018

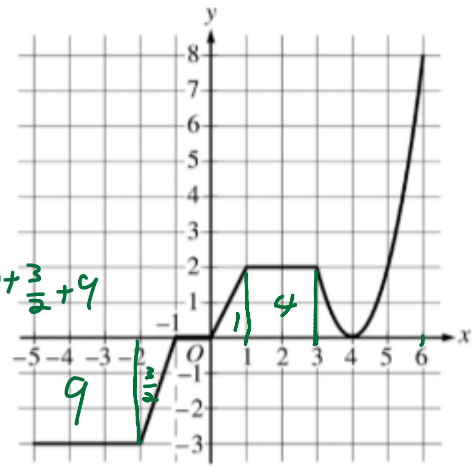
$\int_1^1 F(x)\,dx = 0$

$F'(x) = g(x)$

$\int_1^{-5} F(x)\,dx = -1 + 0 + \frac{3}{2} + 9$

$= 9\frac{1}{2}$

$+3$



Graph of $g$

3. The graph of the continuous function $g$, the derivative of the function $f$, is shown above. The function $g$ is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$ ? $= 12\frac{1}{2} = \frac{25}{2}$

starting point

(b) Evaluate $\int_1^6 g(x)\,dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of $f$ both increasing and concave up? Give a reason for your answer.

(d) Find the $x$-coordinate of each point of inflection of the graph of $f$. Give a reason for your answer.

14 If $y = 2x - 8$, what is the minimum value of the product $xy$?

$2 \cdot 2 - 8 = -4 = y$

$2 \cdot -4$

(A) −16 (B) −8 (C) −4 (D) 0 (E) 2

wrong

$x(2x-8)$

$2x^2 - 8x$

$4x - 8 = 0$

$x = 2$

12 **12.** At what $x$-coordinate does the function $f(x) = 2\sin x + 4x$ have an inflection point on the interval $(0,\ 2\pi)$?

(A) $\frac{\pi}{2}$ (B) $\frac{3\pi}{4}$ (C) $\frac{3\pi}{2}$ (D) $\pi$ (E) $\frac{\pi}{4}$

$F(x) = 2\sin x + 4x$

$F'(x) = 2\cos x + 4$

$F''(x) = 2(-\sin x) + 0$

$0 = -2\sin x$

$\sin x = 0$

$x = 0, \pi, 2\pi$

$-2\sin\frac{\pi}{2}$

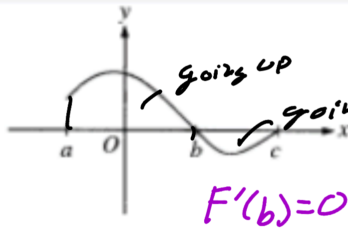
$-2 \cdot 1 = -2$

$-2\sin\frac{3\pi}{2} = -2 \cdot -1 = +2$

$- - - \cdot - \quad \pi \quad + + + + + +$

$\frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2}$

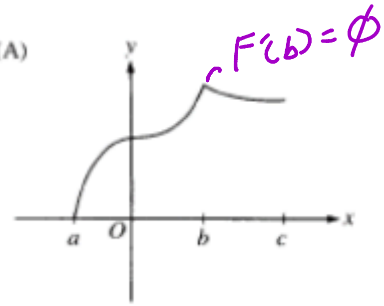
16



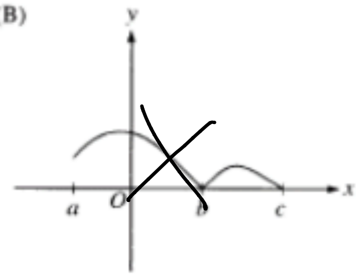

$F'(b)=0$

Let $f(x) = \int_a^x h(t)\,dt$, where $h$ has the graph shown above. Which of the following could be the graph of $f$?

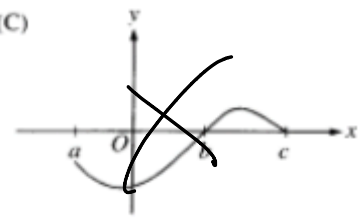
(A)



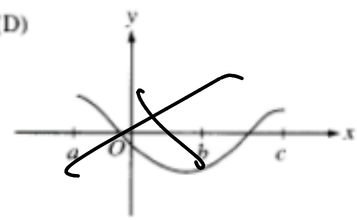

(B)



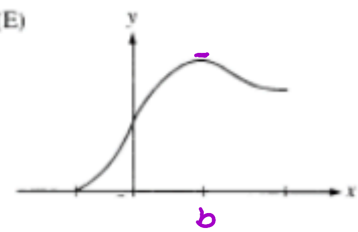

(C)




(D)




(E)



19 If $g$ is a differentiable function such that $g(x) < 0$ (always −) for all real numbers $x$ and if $f'(x) = (x^2 - 4)g(x)$, which of the following is true? B

$0 = (x+2)(x-2)g(x) \Rightarrow x = 2, -2$

(A) $f$ has a relative maximum at $x = -2$ and a relative minimum at $x = 2$.
(B) $f$ has a relative minimum at $x = -2$ and a relative maximum at $x = 2$.
(C) $f$ has relative minima at $x = -2$ and at $x = 2$.
(D) $f$ has relative maxima at $x = -2$ and at $x = 2$.
(E) It cannot be determined if $f$ has any relative extrema.

$F'(-3) = (9-4)g(x) = 5g(x) = 5 \cdot - = -$

$F'(0) = (0-4)g(x) = -4 \cdot g(x) = -4 \cdot - = +$

$F'(3) = (5) \cdot - = -$

− − − − − + + + + + + + + − − −

-3 -2 -1 0 1 2 3

Min Max

From $F'(x) = -$ To $F'(x) = +$

min

From $F'(x) = +$ To $F'(x) = -$

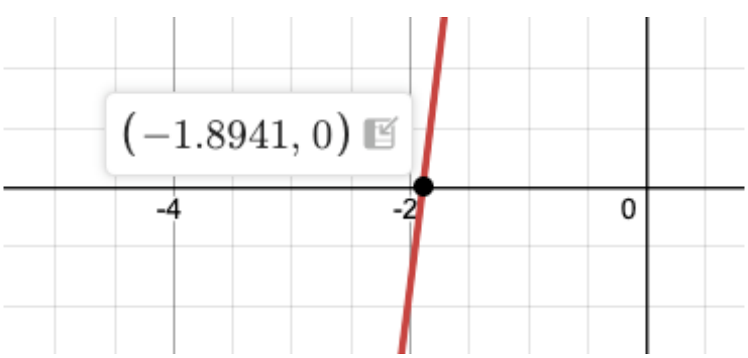
max

13 The graph of the function $y = x^3 + 6x^2 + 7x - 2\cos x$ changes concavity at $x =$

(A) −1.58 (B) −1.63 (C) −1.67 (D) −1.89 (E) −2.33

$$\frac{dy}{dx} = 3x^2 + 12x + 7 + 2\sin x$$

$$\frac{d^2y}{dx^2} = 6x + 12 + 2\cos x = 0$$



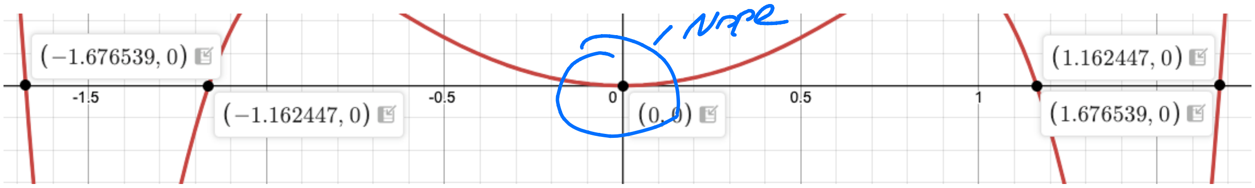

$y = 6x + 12 + 2\cos x$

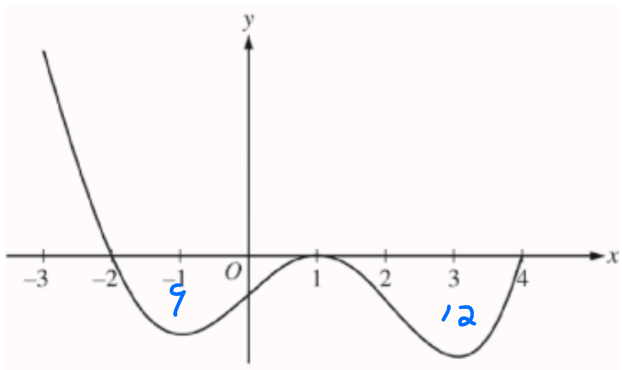
*15. Let $f(x)$ be the function with the derivative defined by $f'(x) = \sin(x^3)$ on the interval $-1.8 < x < 1.8$. How many inflection points does $f(x)$ have on this interval?

(A) two (B) three (C) four (D) five (E) six

$F''(x) = \underbrace{3x^2}_{\text{always } +} \cos(x^3)$

Nope






Graph of $f'$

5. The figure above shows the graph of $f'$, the derivative of a twice-differentiable function $f$, on the interval $[-3, 4]$. The graph of $f'$ has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the $x$-axis and the graph of $f'$ on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

(a) Find all $x$-coordinates at which $f$ has a relative maximum. Give a reason for your answer.

(b) On what open intervals contained in $-3 < x < 4$ is the graph of $f$ both concave down and decreasing? Give a reason for your answer.

(c) Find the $x$-coordinates of all points of inflection for the graph of $f$. Give a reason for your answer.

(d) Given that $\underbrace{f(1)}_{\text{Start}} = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

$$3 + \int_1^x F'(x)dx = F(x)$$

$$3 + \int_1^4 F'(x)dx = -12 + 3 = -9$$

$$3 + \int_1^{-2} F'(x)dx = 3 + 9 = 12$$

15 If the derivative of $f$ is given by $f'(x) = e^x - 3x^2$, at which of the following values of $x$ does $f$ have a relative maximum value?

From + To −

Max of F F(x)

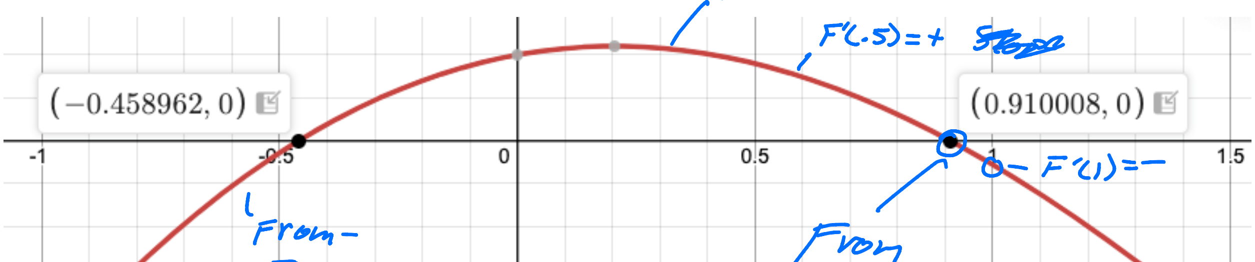
(A) −0.46 (B) 0.20 (C) 0.91 (D) 0.95 (E) 3.73

$F'(x)=0$

$0 = e^x - 3x^2$

$F'(x)$

$F'(.5) = +$ ~~Slope~~




$0 - F'(1) = -$

From − To Positive

min

From Positive slope To negative Slope

max

4. **2003 #83 (AB but suitable for BC) - Calc OK:** The velocity, in ft/sec, of a particle moving along the $x$-axis is given by the function $v(t) = e^t + te^t$. What is the average velocity of the particle from time $t = 0$ to time $t = 3$?

a. 20.086 ft/sec
b. 26.447 ft/seo
c. 32.809 ft/sec
d. 40.671 ft/sec
e. 79.342 ft/sec

$$\frac{1}{b-a}\int_a^b v(T)\,dT = \frac{1}{3}\cdot\int_0^3 \left(e^T + Te^T\right)dT$$

$$y = \int_0^3 \left(e^x + xe^x\right)dx$$

$= 60.2566107696$

$$\frac{1}{3}(60.2566)$$